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**STOCHASTIC PROCESS APPROXIMATION FOR RECURSIVE
ESTIMATION WITH GUARANTEED BOUND ON THE
ERROR COVARIANCE**

Giuseppe Menga

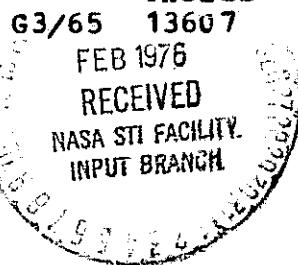
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STOCHASTIC PROCESS APPROXIMATION FOR RECURSIVE
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ABSTRACT

A new approach is proposed for the design of approximate, fixed-order, discrete time realizations of stochastic processes from the output covariance $\mathcal{R}(i,j)$ over a finite time interval. No restrictive assumptions are imposed on the process; it can be nonstationary and lead to a high dimension realization. Classes of fixed-order models are defined, having the joint covariance matrix of the combined vector of the outputs in the interval of definition greater or equal than the process covariance (the difference matrix is non-negative definite). The design is achieved by minimizing, in one of those classes, a measure of the approximation between the model and the process evaluated by the trace of the difference of the respective covariance matrices. The models belonging to these classes have the notable property that, under the same measurement system and estimator structure, the output estimation error covariance matrix computed on the model is an upper bound of the corresponding covariance on the real process.

An application of the approach is illustrated by the modeling of random meteorelogical wind profiles from the statistical analysis of historical data.

I. INTRODUCTION

Given an m -vector-valued discrete-time process $y(\cdot)$ over a finite interval $[1,N]$, with a positive definite covariance $\mathcal{R}(i,j)$, we consider the problem of computing a markovian representation, with output $y_M(i)$, which statistically approximates $y(i)$ in the interval $1 \leq i \leq N$ of the form

$$\left. \begin{aligned} x(i+1) &= A(i) \cdot x(i) + B(i) \cdot w(i) \\ y_M(i) &= C(i) \cdot x(i) + D(i) \cdot w(i) \end{aligned} \right\} \quad (1.1)$$

where the dimensions of $x(i)$, $w(i)$, $A(i)$, $B(i)$, $C(i)$, $D(i)$ ¹ are $n \times 1$, $p \times 1$, $n \times m$, $n \times p$, $m \times m$, $m \times p$, respectively, and $w(i)$ is a p -vector-valued independent

¹The matrices $D(i)$ can also be assumed to equal zero.

noise. The dimension p of $w(\cdot)$ and the order n of the model are chosen by the designer, noting that the order of the model may be lower than the exact minimal realization of the process. Neither stationarity nor the hypothesis on the order of the minimal realization of the process is postulated.

The model is used to design and evaluate linear recursive estimators of the process output in the interval $1 \leq i \leq N$ - that is, the process output will be observed through a linear measurement system with independent additive noise. The measurement system and, in turn, the filter characteristics are not actually specified. However, for the same arbitrary measurement system and estimator, the output estimation error covariance matrix² computed on the model from measurements in the interval $1, \dots, i$ ($1 \leq i \leq N$) is required to be greater than or equal to the equivalent estimation covariance on the real process, that is, the difference matrix is nonnegative. The models with this property are indicated here as "statistically guaranteed models" of the process.

General solutions of the exact or partial realization (specifically minimal realization) problems exist for stationary processes with a lumped representation (refs. 1-5). In contrast, dealing with nonstationary processes, the literature commonly assumes that an analytic expression of the output covariance is available in a separable form (refs. 6-9). This form also implies a lumped realization.

Therefore, if only the numerical values of the covariance are given, the computation of a separable form, when it exists, is preliminarily required. This approach, though theoretically viable, does not work out the practical difficulties encountered when the resulting realization is of high dimension or of changing structure in time. Dealing with an experimental output covariance which does not satisfy precise analytical assumptions, it may be necessary to adopt a lower-order approximation in several practical applications. However it seems that no³ general results are available to the approximate realization of an arbitrary process with a constrained-order model.

The approximation technique proposed here is to minimize in a class of "guaranteed models" of fixed order n a measure of the approximation between model and process evaluated by the trace of the difference between the respective covariance matrices (a partial minimization scheme is proposed in the paper). It will be shown that guaranteed models are characterized by having an output covariance matrix greater than or equal to the process covariance.⁴ This property offers, among other advantages, the option to use the relatively simple performance measure that we have proposed. It is evident that computational limitations exist if N is large and no restrictive assumptions are taken for the process. It is shown, however, that the general approach can be

²The joint covariance matrix of the combined vector of filtering and prediction output errors from i to N .

³For an example of approximation of stationary processes by ARMA models, see reference 10.

⁴A similar property in a less general approximation approach was already mentioned in reference 11.

applied with simplifications to stationary processes with finite dimension realization even if N is large.

The probabilistic model of vertical profile of atmospheric wind velocity from statistics of historical records of wind measurements is a case of non-stationary, high-order realization⁵ process and it is presented here as an example of the method. The model has been used in reference 12 to estimate the onboard wind shear estimation from an airplane in descent flight.

II. GUARANTEED COVARIANCE ESTIMATION

Consider the estimation at instant i ($1 \leq i \leq N$), with a linear estimator, of the present and future portion of the process $y(j)$ ($j = i, \dots, N$) from the present and past measurements operated on the process output through a linear measurement system with additive independent noise. Referring to past portions of the process output, define the combined vector

$$\underline{y}(i) = [y'(i), y'(i-1), \dots, y'(1)]' \quad ^6 \quad (2.1)$$

and the joint covariance matrix

$$P(i) = \begin{bmatrix} R(i,i), \dots, R(i,1) \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ R(1,i), \dots, R(1,1) \end{bmatrix} \quad 1 \leq i \leq N \quad (2.2)$$

with a similar notation defined for the model $\underline{y}_M(i)$ and $P_M(i)$. The present and future portion of the process is indicated as

$$\mathcal{C}_2 \underline{y}(N) \quad (2.3)$$

where \mathcal{C}_2 is a matrix $(N - i + 1)m \times Nm$ of the form

$$\mathcal{C}_2 = [I(N - i + 1)m | 0] \quad (2.4)$$

and its estimate, by a linear estimator \mathcal{K} , is

$$\mathcal{C}_2 \underline{y}(N) = \mathcal{K} \cdot Z \quad (2.5)$$

The vector Z of dimension $m \cdot i$ is the present and past noisy measurement of $\underline{y}(N)$

$$Z = \mathcal{C}_1 \underline{y}(N) + V \quad (2.6)$$

⁵The high dimensionality of the realization is with respect to the total number of samples N of the process. This makes impractical a recursive estimation.

⁶The prime ' indicates the transpose.

The matrix $\mathcal{E}_1 [im \times Nm]$ of full rank $i - m$ is of the form

$$\mathcal{E}_1 = [0; \Pi] \quad (2.7)$$

where Π of dimension $im \times im$ accounts for a possible dynamical structure of the measurement system. The vector V is a $(i - m)$ -dimensional zero mean independent measurement noise of covariance R . The output estimation error covariance of the vectors (2.3), indicated by $\hat{P}(N,i)$, resulting from the estimator \mathcal{K} of Eq. (2.5) and the measurement system of Eqs. (2.6-2.7) is

$$\hat{P}(N,i) = (\mathcal{E}_2 - \mathcal{K} \cdot \mathcal{E}_1) \cdot P(N) \cdot (\mathcal{E}_2 - \mathcal{K} \cdot \mathcal{E}_1)' + \mathcal{K} \cdot R \cdot \mathcal{K}' \quad (2.8)$$

With the above notations, the following theorem is proven.

Theorem 1: Given two processes of the form of (2.1) having covariance matrices (2.2) of values $P_1(N)$ and $P_2(N)$, respectively, and for the same arbitrary measurement system and estimator, as given in Eqs. (2.5) and (2.6), a necessary and sufficient condition for the respective estimation covariance matrices (2.8) to satisfy

$$\hat{P}_1(N,i) - \hat{P}_2(N,i) \geq 0 \quad 1 \leq i \leq N \quad (2.9)$$

is

$$P_1(N) - P_2(N) \geq 0 \quad (2.10)$$

Proof: Sufficient condition is immediate consequence of the quadratic form (2.11)

$$\hat{P}_1(N,i) - \hat{P}_2(N,i) = \hat{\Delta}(N,i) = (\mathcal{E}_2 - \mathcal{K} \cdot \mathcal{E}_1) \cdot \Delta(N) \cdot (\mathcal{E}_2 - \mathcal{K} \cdot \mathcal{E}_1)' \quad (2.11)$$

where

$$\Delta(N) = P_1(N) - P_2(N)$$

To prove the necessity part, expression (2.11) is rewritten as

$$\hat{\Delta}(N,i) = (I - \mathcal{K}) \cdot \begin{vmatrix} \mathcal{E}_2 \\ \mathcal{E}_1 \end{vmatrix} \cdot \Delta(N) \cdot (\mathcal{E}_2' \mathcal{E}_1') \cdot \begin{vmatrix} I \\ -\mathcal{K}' \end{vmatrix} \quad (2.12)$$

Then it is observed that the rows of $\begin{vmatrix} \mathcal{E}_2 \\ \mathcal{E}_1 \end{vmatrix}$ span the full $(N + m)$ -dimension space of $\Delta(N)$ and it is recalled that the rows of the matrix \mathcal{K} are arbitrary.

As a result of theorem 2 it follows that any model realization (1.1) having output covariance matrix $P_M(N)$ greater than or equal to the process covariance $P(N)$ generates, for the same arbitrary linear estimator and linear measurement system with additive independent noise, an upper bound of the process estimation error covariance ($\hat{P}_M(N,i) = \hat{P}(N,i) \geq 0$) in the interval $1 \leq i \leq N$.

When the measurement characteristics become available, the Kalman filter computed for the best⁷ model in a class with the previous properties is a reasonable choice for the process output estimation. Such a filter guarantees the minimum (with reference to the chosen model) upper bound of the covariance of the estimation error, and it also prevents the divergence of the estimates. It is well known that this phenomenon may occur when, because of model approximations, the calculated estimation covariance becomes unrealistically small with respect to the real one on the process (ref. 14).

The whole optimum filter, with reference to the entire model class, also could be sought. However this problem requires the computation of the limit of a sequence of models M_i in the class having covariance

$$P(N) \leq P_{M_1}(N) \leq \dots \leq P_{M_0}(N)$$

and this has not been considered here.

III. GUARANTEED MODEL REALIZATION

The computation of exact or partial realizations on the form of (1.1) from the output covariance $\mathcal{R}(i,j)$ is essentially equivalent to the algebraic problem to find a decomposition for the covariance matrix

$$P(N) = T(N) \cdot \Lambda(N) \cdot T'(N) \quad (3.1)$$

where $P(N)$ has been defined in (2.2),

$$\Lambda(N) = \begin{bmatrix} \tilde{S}(N-1) & & \\ & \ddots & \\ & & \tilde{S}(0) \end{bmatrix} \quad (3.2)$$

is a symmetrical block diagonal matrix with $\tilde{S}(i) >^8 0$ of dimension $m \times m$ and $T(N)$ is a nonsingular causal transformation. If $T(N)$ is also casually invertible then it is called the innovation representation of $y(\cdot)$ and the independent process with covariance Λ the innovation process (ref. 15). Recently, Akaike (refs. 16 and 17) gives of the realization problem an interpretation in terms of canonical regression analysis. His basic idea is adopted here and two regression schemes, the second being the extension of the first, are introduced to generate fixed-order realizations whose output covariance is greater than or equal to the process covariance $P(N)$ (2.2). The starting point of the procedure is the observation that if $y_M(i)$ is the output at instant i of a finite dimension model, then it can be expressed by

⁷The best model measured by the trace $(P_M(N) - P(N))$.

⁸Strictly inequality because of our assumption that $P(N)$ positive definite.

$$y_M(i) = \bar{K}(i-1) \cdot \begin{vmatrix} y_M(i-1) \\ \vdots \\ y_M(i-r) \end{vmatrix} + u_{\text{res}}(i-1) \quad (3.3)$$

where $\bar{K}(i-1)$ is of dimension $m \times r \cdot m$ and r is a finite integer. The residual vector $u_{\text{res}}(i-1)$, uncorrelated with the outputs prior to $i-r$, may be in general correlated with $(y_M'(i-1), \dots, y_M'(i-r))'$. With respect to the notation proposed in the previous section (2.2) indicate with $P(i)$ and $P_M(i)$ the portions of covariance matrices of process and model in the interval $i, i-1, \dots, 1$ and with $\Delta(i)$ the difference matrix

$$P_M(i) - P(i) = \Delta(i) \quad (3.4)$$

Introduce for the process covariance the partition

$$P(i) = \begin{bmatrix} P_{11}(i-1) & | & P_{12}(i-1) \\ \hline \cdots & | & \cdots \\ P_{12}'(i-1) & | & P(i-1) \end{bmatrix} \quad (3.5)$$

where $P_{11}(i-1) = R(i,i)$ and $P_{12}(i-1) = (R(i, i-1), \dots, R(i, 1))'$ are of dimension $m \times m$ and $m \times (i-1) \cdot m$. Choose the model order $n = m \cdot r^9$ and impose that the first s samples $(y_M(i-1), \dots, y_M(i-s))'$ correlate with the residual. From expression (3.3) and related arguments, the model output covariance $P_M(i)$ can be decomposed in

$$P_M(i) = \begin{bmatrix} I_m & | & K(i-1) \\ \hline \cdots & | & \cdots \\ 0 & | & I_m(i-1) \end{bmatrix} \cdot \begin{bmatrix} S(i-1) & | & R(i-1) \\ \hline \cdots & | & \cdots \\ R'(i-1) & | & P_M(i-1) \end{bmatrix} \begin{bmatrix} I_m & | & K(i-1) \\ \hline \cdots & | & \cdots \\ 0 & | & I_m(i-1) \end{bmatrix}' \quad (3.6)$$

where $K(i-1)$ and $R(i-1)$ are both matrices of dimension $m \times (i-1) \cdot m$ and are constrained, for all $i: m(i-1) \geq n$ to the form

$$\left. \begin{aligned} K(i-1) &= [\bar{K}(i-1) \mid 0] \\ R(i-1) &= [\bar{R}(i-1) \mid 0]. \end{aligned} \right\} \quad (3.7)$$

The matrix $\bar{K}(i-1)[m \times n]$ has already been defined in Eq. (3.3), the matrix $\bar{R}(i-1)[m \times ms]$, $ms \leq n$ accounts for the nonzero correlation terms of the residual vector $u_{\text{res}}(i-1)$ in (3.3) with $(y_M'(i-1), \dots, y_M'(i-r))'$ and $S(i-1)[m \times m]$ is the covariance of the residual $u_{\text{res}}(i-1)$.

⁹Constraining the model order to be a multiple of the number of outputs is imposed for notation simplicity, but it is not strictly necessary.

The model covariance $P_M(N)$ is therefore completely defined by a sequence of decompositions (3.6) or, in alternate form, with the introduction of a new sequence of covariance matrices $\tilde{V}(i) [m \cdot i \times m \cdot i]$, by

$$\begin{aligned} \tilde{V}(i) &= \begin{bmatrix} I_m + K(i-1) \cdot \tilde{R}'(i-1) \cdot \tilde{S}^{-1}(i-1) & | & K(i-1) \\ \hline -\tilde{R}'(i-1) \cdot \tilde{S}^{-1}(i-1) & | & I_m(i-1) \end{bmatrix} \cdot \begin{bmatrix} \tilde{S}(i-1) & | & 0 \\ \hline 0 & | & \tilde{V}(i-1) \end{bmatrix} \\ &\cdot \begin{bmatrix} I_m + K(i-1) \cdot \tilde{R}'(i-1) \cdot \tilde{S}^{-1}(i-1) & | & K(i-1) \\ \hline -\tilde{R}'(i-1) \cdot \tilde{S}^{-1}(i-1) & | & I_m(i-1) \end{bmatrix} \end{aligned} \quad (3.8)$$

and

$$\tilde{V}(i-1) = \tilde{P}_M(i-1) - [\tilde{R}' \cdot \tilde{S}^{-1} \cdot \tilde{R}]_{(i-1)}$$

where the matrices $\tilde{P}_M(i-1)$, $\tilde{R}(i-1)$, $\tilde{S}(i-1)$ are derived from (3.6) with the following algorithm

$$\begin{bmatrix} \tilde{S}(i-1) & | & \tilde{R}(i-1) \\ \hline -\tilde{R}'(i-1) & | & \tilde{P}_M(i-1) \end{bmatrix} = \begin{bmatrix} I_m & | & -K(i-1) \\ \hline 0 & | & I_m(i-1) \end{bmatrix} \cdot [\tilde{P}_M - \tilde{R}' \cdot \tilde{S}^{-1} \cdot \tilde{R}]_{(i)} \cdot \begin{bmatrix} I_m & | & -K(i-1) \\ \hline 0 & | & I_m(i-1) \end{bmatrix} \quad (3.9)$$

starting from

$$\tilde{P}_M(N) = \tilde{V}(N) = P_M(N), \text{ and } [\tilde{R}' \cdot \tilde{S}^{-1} \cdot \tilde{R}]_{(N)} = 0.$$

Note also that the matrices $\tilde{R}(i)$ and $\tilde{P}_M(i)$ have structure

$$\begin{aligned} \tilde{R}(i) &= \left[\begin{array}{c|c} \tilde{R}(i) [m \times ms] & | & 0 \\ \hline \end{array} \right] \\ \tilde{P}_M(i) &= P_M(i) - \left[\begin{array}{c|c} Q(i) [m \times (s-1) \times m \times (s-1)] & | & 0 \\ \hline 0 & | & 0 \end{array} \right] \end{aligned} \quad (3.10)$$

for certain matrices $\tilde{R}(i)$ and $Q(i)$.

With the above assumptions the following theorem is proven.

Theorem 2: The model representation with output covariance

$$P_M(N) = T(N) \cdot \Lambda(N) \cdot T'(N) \quad (3.11)$$

where $\Lambda(N)$ has been defined in (3.2) with $\tilde{S}(i)$ given from (3.9) and the transformation $T(N) [N \cdot m \times N \cdot m]$ given by

$$T(N) = [t(N-1)] \cdot \begin{bmatrix} I_m & 0 \\ 0 & t(N-2) \end{bmatrix} \cdots \begin{bmatrix} I_m(N-1) & 0 \\ 0 & t(1) \end{bmatrix}$$

with $t(i)$ of the form

$$t(i) = \begin{bmatrix} I_m + K(i) \cdot R'(i) \cdot S^{-1}(i)' K(i) \\ R'(i) \cdot S^{-1}(i) \\ I_m \cdot i \end{bmatrix}$$

guarantees an upper bound of the process output covariance matrix:

$$P_M(N) - P(N) \geq 0$$

if the matrices $S(i)$ defined in (3.6) are chosen as

$$\begin{aligned} S(0) &= \Delta_S(0) + P(1) \\ S(i) &= \Delta_S(i) + P_{11}(i) + [R(i) + K(i) \cdot P_M(i) - P_{12}(i)] \cdot \Delta^{-1}(i) \\ &\quad \cdot [R(i) + K(i) \cdot P_M(i) - P_{12}(i)]' - K(i) \cdot R'(i) \\ &\quad - R(i) \cdot K'(i) - K(i) \cdot P_M(i) \cdot K'(i) \quad \forall i, \quad i = 1, \dots, N-1 \end{aligned} \tag{3.12}$$

for arbitrary symmetrical matrices $\Delta_S(i) [m \times m]$ satisfying

$$\Delta_S(i) > 0, \quad i = 0, \dots, N-2, \quad \Delta_S(N-1) \geq 0.$$

Proof: Observe first that the representation (3.11) is nothing but the recursive inclusions of expressions (3.8) and therefore equivalent and deducable from (3.6). Then decompose $P_M(N)$ and $P(N)$ according to (3.5) and (3.6) ($i = N$). The result yields

$$\begin{bmatrix} [S+K \cdot R'+R \cdot K'+K \cdot P_M \cdot K']_{(N-1)} & [R+K \cdot P_M]_{(N-1)} \\ [R+K \cdot P_M]'_{(N-1)} & P_M(N-1) \end{bmatrix} - \begin{bmatrix} P_{11}(N-1) & P_{12}(N-1) \\ P_{12}'(N-1) & P(N-1) \end{bmatrix} = \Delta(N) \geq 0 \tag{3.13}$$

Inequality (3.13) is satisfied if

$$P_M(N-1) - P(N-1) = \Delta(N-1) > 0 \tag{3.14}$$

and

$$S+K \cdot R'+R \cdot K'+K \cdot P_M \cdot K'-P_{11}-[R+K \cdot P_M-P_{12}] \cdot \Delta^{-1} \cdot [R+K \cdot P_M-P_{12}]' \Big|_{N-1} = \Delta_S(N-1) \geq 0 \tag{3.15}$$

Proceeding backward, inequality (3.14) is successively partitioned with use of (3.6) and replaced by conditions of the form of (3.15) (the \geq is replaced by $>$). This proves the theorem.

It remains to show that (3.11) is also a causal representation of $y_M(i)$ for $1 \leq i \leq N$ and it can be reduced to a model representation of the form (1.1). This is easily accomplished, defining an m -valued-vector process $(x'(i+s), x'(i+s-1), \dots, x'(1))'$ with covariance matrix

$$T(i+s) \cdot \Lambda(i+s) \cdot T'(i+s) \\ = [t(i+s-1)] \dots \Lambda(i+s) \dots [t(i+s-1)]' = \tilde{V}(i+s).$$

Then, from recursion (3.9) and (3.10), it is observed that the lowest $i \cdot m$ portion of $x(\cdot)$ is statistically equivalent to the corresponding $y_M(\cdot)$ process from 1 to i .¹⁰ A proper restriction of $x(\cdot)$ with a formal shift of the index i of s steps on time is therefore the required realization. Note also that the remaining components $x(j)$, $j = i+1, \dots, i+s$ are the predictors of the future $y_M(\cdot)$ from observations until i according to the theory developed by Akaike (ref. 17).

The model realization is finally defined by the following matrices. These are deducible from Eqs. (3.8) and (3.9).

$$A(i) = \begin{bmatrix} \bar{K}(i+a) \\ \cdots \\ 1 \\ \cdots \\ 1 & 0 \end{bmatrix}^{11} \quad (3.16)$$

$$B(i)[n \times m] = \begin{bmatrix} I_m + K \cdot \tilde{R}' \cdot \tilde{S}^{-1} \\ \cdots \\ U_p(\tilde{R}' \cdot \tilde{S}^{-1}) \end{bmatrix}_{i+a} \quad (3.17)$$

$$C(i) = (0_1, \dots, 0_a, I_m, 0) \quad (3.18)$$

$$D(i) = \text{Low}(\tilde{R}' \cdot \tilde{S}^{-1}) \Big|_{i+a} \quad (3.19)$$

$$\text{cov}(w(i)) = \tilde{S}(i+a) \quad (3.20)$$

$$\text{cov}(x_1) = \begin{bmatrix} \tilde{P}_M - \tilde{R}' \cdot \tilde{S}^{-1} \cdot \tilde{R} & | & 0 \\ \cdots & | & \cdots \\ 0 & | & 0 \end{bmatrix}_{1+a} \quad (3.21)$$

¹⁰If $s = r$, a noise vector has to be added to $x(\cdot)$. This originates strictly a nonproper model realization (see eq. (3.19)).

¹¹In the case that $s \cdot m < n - m$ zero matrices complete $\bar{K}(i)$ for the first values of i .

where $U_p(\cdot)$ and $Low(\cdot)$ are respectively operators which isolate the first $n - m$ and following m rows of the matrix (\cdot) . The dimension p of the input process $w(i)$ results here in $p = m$. The index a appearing in (3.16)-(3.21) is $a = r - 1$ if $s \cdot m = n$, and $a = s$ if $s \cdot m \leq n - m$; in this case, the matrices $D(i)$ are void.

Because of the generality of the assumptions, structures of the form of (3.6) can represent with a finite value of r (and s) arbitrary linear lumped systems. However they do not encompass the whole class of the finite realizations, and in fact, for a given model covariance the result may not be the minimal realization. For stationary processes, analysis of this point has been at the basis of the Ho's algorithm (ref. 1). For nonstationary processes, the problem received a theoretical interpretation from canonical regression analysis (ref. 17). Justified from the results of canonical regression analysis, an extension of our previous result is given next.

The approach requires in this case to fix the model order n (dimension of the space spanned by the projection of future outputs on the past outputs) and the number r ($m \cdot r > n$) of output samples having independent projection. Then at each instant i with a nonsingular transformation an n -dimensional basis, spanning the projection space is isolated from the whole $m \cdot r$ vector of outputs $(y_M'(i-1), \dots, y_M'(i-r))'$. Indicate this transformation with

$$L(i) = \begin{bmatrix} L_1(i) & | & L_2(i) & | & \dots & | & 0 \\ \hline 0 & | & I_m(i-r) & | & & & \end{bmatrix} \quad (3.22)$$

$$L_1(\cdot) [m \cdot r \times (m \cdot r - n)], \quad L_2(\cdot) [m \cdot r \times n].$$

The model covariance assumes the following structure

$$P_M(i) = L(i) \cdot H(i) \cdot L'(i) \quad (3.23)$$

with

$$H(i) = \begin{bmatrix} H_{11}(i) & | & 0 \\ \hline 0 & | & H_{12}(i) \\ \hline 0 & | & H_{12}'(i) & | & P_M(i-r) \end{bmatrix} \quad (3.24)$$

$$H_{11}(\cdot) [m \cdot r \times m \cdot r], \quad H_{12}(\cdot) [n \times (i-r) \cdot m]$$

The form (structure) of $H(\cdot)$ is explained from the fact that at every instant i only n independent components of the whole $m \cdot r$ output samples space correlate with the past.

Combining the transformation (3.23) with the previous decomposition (3.6) results in

$$P_M(i) = L(i) \cdot H(i) \cdot L'(i) = \begin{bmatrix} I_m & K(i-1) \\ 0 & L(i-1) \end{bmatrix} \cdot \begin{bmatrix} S(i-1) & R(i-1) \\ R'(i-1) & H(i-1) \end{bmatrix} \cdot \begin{bmatrix} I_m & K(i-1) \\ 0 & L(i-1) \end{bmatrix} \quad (3.25)$$

The recursivity of (3.25) imposes the following constraint (3.26) (this is derived by substituting (3.22) and (3.24) in (3.25) and developing the products)

$$\begin{bmatrix} (\bar{K} + \bar{R} \cdot H_{11}) \cdot \begin{vmatrix} L'_{21} \\ L'_{22} \end{vmatrix} \mid \bar{K}_2 \cdot H_{12} \\ \hline (L_{11} \mid L_{12}) \cdot H_{11} \cdot \begin{vmatrix} L'_{21} \\ L'_{22} \end{vmatrix} \mid L_{12} \cdot H_{12} \end{bmatrix}_{i-1} = L_2(i) \cdot H_{12}(i) \quad (3.26)$$

with matrices of dimensions

$$\bar{K}(\cdot) [m \times m \times r], \quad \bar{R}(\cdot) [m \times m \times r]$$

$$(L_1(i-1) \mid L_2(i-1)) = \begin{bmatrix} L_{11}(i-1) & L_{12}(i-1) \\ \hline L_{21}(i-1) & L_{22}(i-1) \end{bmatrix}$$

$$\bar{K}(i-1) = (\bar{K}_1(i-1) \mid \bar{K}_2(i-1))$$

$$L_{11}(\cdot) [(m+r-m) \times (m+r-n)], \quad L_{12}(\cdot) [(m+r-m) \times n],$$

$$L_{21}(\cdot) [m \times (m+r-n)], \quad L_{22}(\cdot) [m \times n],$$

$$\bar{K}_1(\cdot) [m \times (m+r-n)], \quad \bar{K}_2(\cdot) [m \times n].$$

The condition (3.26) is equivalent (up to a nonessential arbitrary transformation)

$$L_2(i) = \begin{bmatrix} \bar{K}_2(i-1) \\ \hline L_{12}(i-1) \end{bmatrix} \quad (3.27)$$

$$\begin{bmatrix} \bar{R} + \bar{K} \cdot H_{11} \\ \hline (L_{11} \mid L_{12}) \cdot H_{11} \end{bmatrix} \cdot \begin{bmatrix} L'_{21} \\ L'_{22} \end{bmatrix}_{i-1} = \begin{bmatrix} \bar{K}_2(i-1) \\ \hline L_{12}(i-1) \end{bmatrix} \cdot h_{12}(i-1) \quad (3.28)$$

$$h_{12}(i-1) [n \times m]$$

and

$$H_{12}(i) = [h_{12}(i-1) | H_{12}(i-1)]. \quad (3.29)$$

With the formal substitutions of

$$K + P_M + K' \Big|_i \quad \text{with} \quad K + H + K' \Big|_i$$

and

$$R + K + P_M \Big|_i \quad \text{with} \quad \left[\begin{array}{c|c} (\bar{R} + \bar{K} + H_{11}) & \begin{matrix} L'_1 \\ L'_2 \end{matrix} \\ \hline L'_1 & \bar{K}_2 + H_{12} \end{array} \right] \Big|_i \quad (3.30)$$

Theorem 2 is extended to realizations of the form of (3.25). From the appendix, where a recursion similar to (3.8) is derived for the general case, it appears that the matrices $t(i)$ in (3.11) are now

$$t(i) = \left[\begin{array}{c|c} I_m & K + \tilde{R}' + \tilde{S}^{-1} \\ \hline L + \tilde{R}' + \tilde{S}^{-1} & I_{m \cdot i} \end{array} \right]_i \quad (3.31)$$

From the appendix, also, the model matrices can be computed and they result in

$$A(i) = I_n \quad (3.32)$$

$$B(i) [n \times (m(r+i)-n)] = [(-\bar{V}'_{12} \cdot \bar{V}_{11}^{-1} | I_n) \cdot (L_1 | L_2)^{-1}] \Big|_{i+r}$$

$$\cdot \left[\begin{array}{c|c} I_m + K + R' + S^{-1} & \begin{matrix} \bar{K} \\ \vdots \\ \bar{K} \end{matrix} \\ \hline (L_{11} | L_{12}) \cdot \tilde{R}' + \tilde{S}^{-1} & \begin{matrix} L_{11} \\ L_{12} \end{matrix} \end{array} \right] \cdot \left[\begin{array}{c|c} I_{m(r+1)-n} \\ \hline \bar{V}'_{12} \cdot \bar{V}_{11}^{-1} \end{array} \right]_{(i+r-1)} \quad (3.33)$$

$$C(i) = L_{22}(i+r-1) \quad (3.34)$$

$$D(i) [m \times m(r+1) - n] = (L_{21} | L_{22}) \cdot \left[\begin{array}{c|c} \tilde{R}' + \tilde{S}^{-1} & \begin{matrix} I_{m(r+1)-n} \\ \hline \bar{V}'_{12} \cdot \bar{V}_{11}^{-1} \end{matrix} \end{array} \right]_{i+r-1} \quad (3.35)$$

$$\text{cov}(w(i)) = \begin{bmatrix} \tilde{s}(i+r-1) & 0 \\ 0 & \bar{V}_{11}(i+r-1) \end{bmatrix} \quad (3.36)$$

$$\text{cov}(x(l)) = [\bar{V}_{22} - \bar{V}_{12}' \cdot \bar{V}_{11}^{-1} \cdot \bar{V}_{12}] \Big|_r \quad (3.37)$$

IV. PROCESS APPROXIMATION

Defined by the order n , the number $m \cdot s$ of the columns of the matrices $\bar{R}(\cdot)$ or the number of independent output samples r , classes of "guaranteed models" are spanned by the choice of matrices $K(\cdot)$, $R(\cdot)$, and $\Delta_S(\cdot) \geq 0$. The proof of theorem 2 offers a constructive sequential algorithm to generate model realizations. Models in these classes are general enough to encompass finite realizations or, in the second case, minimal realizations to every linear causal lumped process. Since we are interested here in the problem of the approximate realization, a measure of the approximation between models in the class and the process is defined by

$$J = \text{trace}(P_M(N) - P(N)) \quad (4.1)$$

The function J is, only for models in the class, an upper bound of the linear operator's norm from the Euclidean space $(N \cdot m)$ into itself, and is zero when $P_M(N) = P(N)$

$$J \geq \|P_M(N) - P(N)\| = \lambda_{\max}(P_M(N) - P(N)) \quad (4.2)$$

where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of the argument matrix. With the use of the partitions (3.5), (3.6) and of the relation (3.14), the function J is given in the form

$$J = \sum_{i=1}^{N-1} \text{trace} \left\{ [\Delta_S(i) + [(R+K \cdot P_M - P_{12}) \cdot \Delta^{-1} \cdot (R+K \cdot P_M - P_{12})'] \Big|_i + \Delta_S(0)] \right\} \quad (4.3)$$

where $P_M(i)$ is generated from the recursion relationship given by (3.6) and $\Delta^{-1}(i)$ from the following:

$$\Delta^{-1}(i) =$$

$$\begin{bmatrix} \Delta_S^{-1} & [-\Delta_S^{-1} \cdot (R+K \cdot P_M - P_{12}) \cdot \Delta^{-1}] \\ [-\Delta_S^{-1} \cdot (R+K \cdot P_M - P_{12}) \cdot \Delta^{-1}] & [\Delta^{-1} + \Delta^{-1} \cdot (R+K \cdot P_M - P_{12})' \cdot \Delta_S^{-1} \cdot (R+K \cdot P_M - P_{12}) \cdot \Delta^{-1}] \end{bmatrix} \Big|_{i-1} \quad (4.4)$$

The model design is achieved by minimizing the approximation measure

$$J^* = \min J$$

with respect to

$$\bar{R}(i), i = 1, N - 1; \quad \bar{K}(i), i = 1, N - 1; \quad \Delta_S(i), i = 0, N - 1$$

with constraints

$$\Delta_S(i) > 0, \quad i = 0, N - 2, \quad \Delta_S(N - 1) \geq 0.$$

A complete minimization of J is impractical, unless the value of N is small. A partial optimization scheme is proposed, which takes advantage of the special sequential structure of the performance index. For an initial guess of matrices, $\Delta_S(i)$ a sequential, term-by-term, optimization with respect to $\bar{K}(i)$ and $\bar{R}(i)$ is performed from $i = 1$ to $i = N - 1$. Since the design matrices appear as a quadratic form on each term of the performance index, an analytic solution is available with use of the matrix gradient relations (ref. 18).¹² Then the procedure is iterated minimizing with numerical search on $\Delta_S(i)$.

Obviously, if no constraining assumptions are taken on the process, storage requirement and computational time limit the maximum size of the interval $1, \dots, N$. We note however that the proposed step-by-step optimization requires the inversion of matrices of fixed dimension ($n \times n$). The problem of the growing dimensionality is found on the few multiplications and sums of matrices required at each step to propagate $P_M(i)$ and $\Delta^{-1}(i)$.

The possibility of extending previous results to an infinite time interval is investigated in the next section for a stationary process with finite dimension realization.

V. APPROXIMATION WITH STATIONARY MODEL

Introduce the partitions defined in (3.5) and (3.6) to the lowest ($m + n$) ($n = m \cdot r$) portions of process and model covariance matrices. Constrain the design model to be stationary. This implies that

$$[S + K \cdot R' + R \cdot K' + K \cdot P_M \cdot K']|_r = UL(P_M(r)) \quad (5.1)$$

$$[R + K \cdot P_M]|_r = [UR(P_M(r))]|_M \quad (5.2)$$

where $UL(\cdot)$, $UR(\cdot)$ are operators which respectively define the partitions of the upper left $m \times m$ and remaining $m \times m$ ($r - 1$) upper right portions of the arguments and where M is an $m \times m$ design matrix. Extend the procedure to $i > m + n$. The conditions given in (3.12) become

$$P_M(r) - P(r) = \Delta(r) > 0 \quad (5.3)$$

¹²Note that in the case of models of the form (3.32-3.38) we adopted $\bar{R}(\cdot)$, and $\bar{K}(\cdot)$ must also satisfy the conditions (3.28).

$$\begin{aligned} \text{UL}(P_M) - P_{11}(r) - [(UR(P_M(r))|_M - P_{12}(r)] \cdot \Delta^{-1}(r) \\ \cdot [(UR(P_M(r))|_M - P_{12}(r)]' = \Delta_S(r) > 0 \end{aligned} \quad (5.4)$$

$$\begin{aligned} \text{UL}(P_M) - P_{11}(i) - [R + K \cdot P_M(i) - P_{12}(i)] \cdot \Delta^{-1} \\ \cdot [R + K \cdot P_M(i) - P_{12}(i)]' = \Delta_S(i) > 0 \end{aligned} \quad (5.5)$$

Note that the index i has been dropped from the matrices independent on i . It is evident from Eqs. (5.1-5.5) that design parameters are $\Delta(r)$, M , and \bar{K} or \bar{R} . However, the approximation measure is a function of the matrix $\Delta(r)$ alone, as the design matrices affect only the values of $\Delta_S(\cdot)$.

$$J = \text{trace} \left\{ (n - r) \cdot \text{UL}(\Delta(r)) + \Delta(r) \right\} \quad (5.6)$$

The performance index is a function of $\Delta(r)$ alone. The other design matrices effect only the values of $\Delta_S(\cdot)$. Therefore, the design is achieved by seeking for the minimum trace matrix $\Delta(r)$, such that it is not, an empty set of values of M and \bar{K} or \bar{R} which satisfy the sequence of constraints

$$\Delta_S(i) > 0, \quad r \leq i \leq N - 2, \quad \Delta_S(N - 1) \geq 0.$$

Assume now to deal with stationary process having finite and stable realization. Expression (5.5) can be written as $\text{UL}(\Delta) - V(i) = \Delta_S(i)$ where $V(i)$ is given by

$$V(i) = UR(\Delta(i + 1)) \cdot \Delta^{-1}(i) \cdot UR(\Delta(i + 1))'$$

The matrix $\Delta(i)$ if stable ensures the existence of "guaranteed model" and is also a portion of the covariance of a stationary process with finite and stable realization. In these conditions Faurre (ref. 2) shows that sequence of matrices of the form $V(i)$ are asymptotically convergent for increasing i .

$$\lim_{i \rightarrow \infty} V(i) = V(\infty)$$

Therefore, a sequence of approximations that are guaranteed on a finite interval i, \dots, N for increasing N , converges asymptotically to a guaranteed approximation in the infinite interval.

VI. EXAMPLE

To illustrate this approach, a stochastic model of the horizontal mean wind velocity, where altitude is taken as an independent parameter, is generated from the statistical analysis of historical data. The model is designed for wind shear prediction from onboard measurements of an aircraft on descent flight (ref. 12).

The wind process is defined as a two-component random vector (southerly and westerly) at intervals of 1 km of altitude from the ground to 10 km. Wind profiles are measured from twice to four times a day by launching meteorological balloons (rawinsonde) from the major airports. Statistical analysis of historical data for the areas of interest is available in the form of monthly or seasonal mean vectors and covariance matrices (ref. 13). A second-order regression scheme of the form of (3.6) with $\bar{R}(i) = 0$ has been chosen to approximate a posteriori covariance of the wind components from 1 to 10 km conditioned to the ground measurements. Actually, the a posteriori covariance of the wind aloft conditioned to the ground measurement has been considered. The procedure proposed in section IV has been pursued constraining for simplicity the sequence $\Delta_S(i)$ to the form

$$\Delta_S(i) = b(i) \cdot I_m$$

and optimizing with numerical search the vector

$$b = (b(0), \dots, b(N - 1))'$$

The process covariance matrix and the resulting model matrices are given in Table 1. The normalized performance index resulted, after minimization, in

$$\text{trace}(P_M(10) - P(10))/\text{trace}(P(10)) = 0.09$$

VII. CONCLUSIONS

A new approach has been proposed in this paper for the design of approximate realizations of stochastic processes from output covariance matrices in a finite interval. This approach does not require any restrictive assumption on the process. It appears also especially convenient in the case of raw experimental covariance.

The novelty of the approach is to define, with simple sequential algorithm elements in classes of fixed-order dynamic models whose output covariance matrix bounds the process covariance in the interval of interest (the difference matrix is not negative definite). The design is achieved by minimizing in a chosen class a measure of the approximation between the model and the process. The specific choice of the classes, which are general enough to encompass finite realizations and minimal realizations of every linear process that admits finite dimension realization, offers two advantages: (1) the possibility to define a relatively simple performance measure (actually the trace of the difference between model and process covariance matrices has been chosen); and (2) models belonging to these classes have the property that with the same output filtering or prediction structure, the estimation error covariance computed on the model is an upper bound of the true estimation covariance on the process. Apart from its intrinsic interest, when the model with such a guaranteed statistic is required for the simulation of the process, this property prevents on estimation the phenomenon of divergence often encountered when approximate models are used for the filter design (ref. 14).

The approach, which for arbitrary process is confined on a finite-time interval, can furnish asymptotically convergent approximations for stationary finite-order realization processes on an infinite interval.

Application is given to the modeling of mean wind randomness as a function of altitude from statistical analysis of historical wind profiles.

APPENDIX

As in the previous case of Eq. (3.8), we define a sequence of covariance matrices $\tilde{V}(i)$, that for convenience is given here in the following transformed form: $\tilde{V}(i) = U(i) \cdot \tilde{V}(i) \cdot U'(i)$, with

$$\tilde{V}(i) = \begin{bmatrix} \bar{V}_{11} & 0 & 0 \\ 0 & \bar{V}_{22} - \bar{V}_{12} \cdot \bar{V}_{11}^{-1} \cdot \bar{V}_{12} & H_{12} \\ 0 & H_{12}' & H_{22} \end{bmatrix}_i$$

$$U(i) = \begin{bmatrix} I_{(m \cdot r - n)} & 0 \\ \bar{V}_{12}' \cdot \bar{V}_{11}^{-1} & I \end{bmatrix}_i$$

$$\bar{V}_{11}(\cdot)[(m \cdot r - n) \times (m \cdot r - n)], \quad \bar{V}_{12}(\cdot)[(m \cdot r - n) \times n], \quad \bar{V}_{22}[n \times n].$$

A recursion equivalent and deducible from (3.25) is

$$L(i) \cdot U(i) \cdot \tilde{V}(i) \cdot U'(i) \cdot L'(i) = \begin{bmatrix} I_m + K \cdot \tilde{R}' \cdot \tilde{S}^{-1} & K \cdot L^{-1} \\ L \cdot \tilde{R}' \cdot \tilde{S}^{-1} & I_{m(i-1)} \end{bmatrix}$$

$$\cdot \begin{bmatrix} \tilde{S} & 0 \\ 0 & L \cdot U \cdot \tilde{V} \cdot U' \cdot L' \end{bmatrix}$$

$$\cdot \begin{bmatrix} I_m + K \cdot \tilde{R}' \cdot \tilde{S}^{-1} & K \cdot L^{-1} \\ L \cdot \tilde{R}' \cdot \tilde{S}^{-1} & I_{m(i-1)} \end{bmatrix}'_{i-1}$$

and

$$[U \cdot \tilde{V} \cdot U']_{i-1} = \tilde{H}(i-1) - [\tilde{R}' \cdot \tilde{S}^{-1} \cdot \tilde{R}]_{(i-1)}$$

where the matrices $\tilde{H}(\cdot)$, $\tilde{R}(\cdot)$, $\tilde{S}(\cdot)$ are derived from (3.25) with the following algorithm

$$\begin{bmatrix} \tilde{S} & \tilde{R} \cdot L' \\ L \cdot \tilde{R}' & L \cdot \tilde{H} \cdot L' \end{bmatrix}_{i-1} = \begin{bmatrix} I_m & -K \cdot L^{-1} \\ 0 & I_{m(i-1)} \end{bmatrix}_{i-1} \cdot [L \cdot (\tilde{H} - \tilde{R}' \cdot \tilde{S}^{-1} \cdot \tilde{R}) \cdot L']_{i-1}$$

$$\cdot \begin{bmatrix} I_m & -K \cdot L^{-1} \\ 0 & I_{m(i-1)} \end{bmatrix}'_{i-1}$$

starting from

$$L(N) + \tilde{H}(N) + L'(N) = P_M(N) \quad \text{and} \quad [\tilde{R}' + \tilde{S}^{-1} + \tilde{R}]_{(N)} = 0.$$

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TABLE I.- WIND COVARIANCE - MODEL MATRICES

A Posteriori Covariance Matrix of the Two Wind Components (Westerly and Southerly) from 10 to 1 km of Altitude,
Conditioned to the Ground Measurements

km	CAPE KENNEDY, FLORIDA												PERIOD OF DATA		
	JANUARY												Jan. 1, 1956 to Nov. 17, 1956		
													Nov. 18, 1956 to Dec. 31, 1956		
1	0.237E 02														
	0.232E 01	0.207E 02													
2	0.195E 02	0.308E 01	0.318E 02												
	0.229E 01	0.206E 02	0.572E 01	0.368E 02											
3	0.172E 02	0.461E 01	0.297E 02	0.893E 01	0.400E 02										
	0.865E 00	0.209E 02	0.392E 01	0.375E 02	0.766E 01	0.524E 02									
4	0.165E 02	0.431E 01	0.267E 02	0.104E 02	0.370E 02	0.104E 02	0.463E 02								
	-0.920E 00	0.212E 02	0.268E 01	0.361E 02	0.604E 01	0.609E 02	0.102E 02	0.619E 02							
5	0.160E 02	0.377E 01	0.280E 02	0.116E 02	0.378E 02	0.127E 02	0.467E 02	0.142E 02	0.604E 02						
	-0.210E 01	0.227E 02	0.170E 01	0.377E 02	0.608E 01	0.526E 02	0.105E 02	0.628E 02	0.156E 02	0.759E 02					
	0.149E 02	0.467E 01	0.285E 02	0.142E 02	0.382E 02	0.153E 02	0.467E 02	0.174E 02	0.604E 02	0.201E 02					
6	0.738E 02														
	-0.293E 01	0.240E 02	0.809E 00	0.403E 02	0.481E 01	0.560E 02	0.868E 01	0.643E 02	0.144E 02	0.773E 02					
	0.187E 02	0.938E 02													
	0.153E 02	0.060E 01	0.297E 02	0.172E 02	0.409E 02	0.197E 02	0.497E 02	0.230E 02	0.636E 02	0.285E 02					
7	0.772E 02	0.281E 02	0.950E 02												
	-0.232E 01	0.235E 02	0.230E 01	0.389E 02	0.727E 01	0.560E 02	0.108E 02	0.646E 02	0.182E 02	0.784E 02					
	0.223E 02	0.844E 02	0.315E 02	0.111E 03											
	0.145E 02	0.613E 01	0.297E 02	0.201E 02	0.419E 02	0.246E 02	0.492E 02	0.281E 02	0.623E 02	0.312E 02					
8	0.756E 02	0.324E 02	0.956E 02	0.379E 02	0.115E 03										
	-0.478E 01	0.237E 02	0.143E 01	0.410E 02	0.737E 01	0.682E 02	0.111E 02	0.670E 02	0.186E 02	0.817E 02					
	0.237E 02	0.974E 02	0.328E 02	0.112E 03	0.405E 02	0.129E 03									
	0.166E 02	0.788E 01	0.319E 02	0.230E 02	0.446E 02	0.296E 02	0.511E 02	0.326E 02	0.642E 02	0.371E 02					
9	0.772E 02	0.394E 02	0.996E 02	0.460E 02	0.120E 03	0.508E 02	0.147E 03								
	-0.588E 01	0.232E 02	0.913E 00	0.398E 02	0.717E 01	0.579E 02	0.966E 01	0.679E 02	0.177E 02	0.842E 02					
	0.231E 02	0.990E 02	0.325E 02	0.112E 03	0.415E 02	0.130E 03	0.505E 02	0.150E 03							
	0.141E 02	0.732E 01	0.289E 02	0.257E 02	0.442E 02	0.315E 02	0.521E 02	0.391E 02	0.666E 02	0.449E 02					
10	0.777E 02	0.490E 02	0.891E 02	0.547E 02	0.118E 03	0.605E 02	0.150E 03	0.619E 02	0.182E 03						
	-0.716E 01	0.204E 02	-0.138E 01	0.361E 02	0.982E 01	0.530E 02	0.776E 01	0.635E 02	0.171E 02	0.785E 02					
	0.227E 02	0.948E 02	0.327E 02	0.108E 03	0.426E 02	0.125E 03	0.504E 02	0.145E 03	0.632E 02	0.163E 03					

Model Matrices from 10 to 1 km of Altitude

B(i) = 1	i	A(i)	cov (w(i))	cov (x _{10})
C(i) = 1				
D(i) = 0	10	0.786D 00 0.526D-01	0.452D-02 0.820D 00	0.344D 02 0.116D 01
	9	0.781D 00 0.627D-01	0.171D-01 0.821D 00	0.236D 02 -0.108D 01
	8	0.823D 00 0.310D-01	-0.973D-02 0.841D 00	0.214D 02 0.483D 00
	7	0.791D 00 0.787D-02	-0.240D-01 0.857D 00	0.151D 02 -0.128D 00
	6	0.828D 00 0.560D-01	-0.129D-01 0.818D 00	0.148D 02 0.222D 00
	5	0.798D 00 0.424D-01	-0.483D-01 0.799D 00	0.111D 02 0.102D 01
	4	0.864D 00 0.405D-01	-0.293D-01 0.829D 00	0.136D 02 0.739D 00
	3	0.736D 00 0.722D-01	-0.683D-01 0.677D 00	0.130D 02 0.219D 00
	2	0.605D 00 -0.124D 00	-0.114D 00 0.582D 00	0.194D 02 0.342D 01
				0.186D 02

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